

# Universal behaviour of four-boson systems from a functional renormalisation group

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We apply a functional renormalisation group to systems of four bosonic atoms close to the unitary limit. We work with a local effective action that includes a trimer field to describe energy dependence of three-body subsystems. We also use this field to eliminate structures that do not correspond to the Faddeev-Yakubovsky equations. In the physical limit, we find three four-body bound states below the shallowest three-body state. The values of the scattering lengths at which two of these states become bound are in good agreement with exact solutions of the four-body equations and experimental observations. The third state is extremely shallow. During the evolution we find an infinite number of four-body states based on each three-body state which follow a double-exponential pattern in the running scale. None of the four-body states shows any evidence of dependence on a four-body parameter.

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Systems where two-body scattering lengths are much longer than ranges of the forces between the particles are important in various areas of physics. Their low-energy properties display universal scaling behaviour, controlled by the “unitary limit” in which the scattering length tends to infinity. In nuclear physics, the large scattering lengths are large enough that low-energy aspects of few-nucleon systems can be described in this framework [1, 2]. In atomic physics, the shallow dimer of  $^4\text{He}$  atoms leads to a scattering length that is about 100 times larger than the size of the atoms [3]. Even better examples are provided by ultra-cold atoms in traps, where Feshbach resonances can be used to tune the scattering lengths to values very close to the unitary limit [4].

In the unitary limit, three-boson systems display a remarkable effect, first predicted by Efimov in 1970 [5, 6]. They possess an infinite tower of three-body bound states, with energies in a constant ratio of 515.0. This breaks the expected scale invariance to a discrete symmetry, with one three-body parameter needed to fix the energies of all these states. In real systems, the sequence of deeply bound states is cut off by the range of the forces, and the shallowest ones by the finite scattering length. Three-fermion systems can also show Efimov behaviour, provided there are enough species to allow spatially symmetric states. Although there were suggestions that the  $A = 3$  nuclei  $^3\text{H}$  and  $^3\text{He}$  could be interpreted as Efimov states [7], the first clear observation of such states was in an ultra-cold gas of caesium atoms [8]. Reviews of the field can be found in Refs. [9–11].

This behaviour in the three-body sector feeds through to four-body systems, where most numerical calculations find two bound states in each Efimov cycle [12–14] whose energies are fixed ratios to the nearest three-body state. However, in contrast, Hadizadeh *et al.* find up to three

four-body states per cycle, with energies that depend on an additional four-body parameter [15, 16], supporting their earlier results of Ref. [17]. Experimental evidence for two four-body states based on an Efimov three-body has been seen in the recombination rates of trapped  $^{133}\text{Cs}$  atoms [18], with resonances that are consistent with the results of Refs. [12–14].

Renormalisation-group methods have been applied to elucidate scaling behaviour in few-body systems [7, 11, 19–21] and hence to determine their relevant parameters. Here we apply a functional renormalisation group (FRG) [23, 24] to the four-boson problem. During the evolution we observe a double-exponential pattern of four-body states built on each three-body state, similar to the “super-Efimov” behaviour found by Moroz, Nishida and Son in a two-dimensional three-body system [22]. These have energies that can be expressed in terms of a universal scaling function, similar to that in Refs. [15, 16], but they show no evidence of dependence on an additional four-body parameter. Away from the unitary limit, we find that three four-body states are present in the last Efimov cycle and so can appear as physical bound states.

The FRG we use is based on a running version of effective action that generates the one-particle irreducible Green’s functions [23, 24]. A regulator is added to the theory to suppress fluctuations with momenta below some scale  $k$ . For large  $k$ , we start with a suitably parametrised “bare” action. In the limit  $k \rightarrow 0$ , all fluctuations are included and we arrive at the physical effective action. Even though it is fully nonperturbative, the driving term in the FRG equation for the action has the form of a one-loop integral. Instead of diagrammatic expansions, practical approximation schemes are obtained by truncating the effective action to a finite number of terms.

This FRG is being applied to systems of nonrelativistic particles, in order to study, in particular, dense matter [25–27]. In that context, it provides an alternative to traditional many-body methods. As part of this programme, studies of few-body systems in the same framework are needed to fix the input parameters. These studies are also proving interesting in their own right [28–31].

In this work we study systems of up to four non-relativistic bosonic “atoms”. We represent the atoms by the field  $\psi(x)$  and we also introduce dimer and trimer fields,  $\phi(x)$  and  $\chi(x)$ , in order to include energy-dependent propagators for two- and three-body subsystems. The evolution equation for the effective action  $\Gamma_k[\psi, \psi^*, \phi, \phi^*, \chi, \chi^*]$  takes the form [24]

$$\partial_k \Gamma = -\frac{i}{2} \text{Tr} \left[ (\partial_k \mathbf{R}) \left( (\Gamma^{(2)} - \mathbf{R})^{-1} \right) \right] + \frac{\delta \Gamma}{\delta \Phi} \cdot \partial_k \Phi, \quad (1)$$

where  $\Gamma^{(2)}$  denotes the matrix of second derivatives of the action with respect to the fields and  $\mathbf{R}$  the regulator that

is added to suppress low-momentum modes. The trace  $\text{Tr}$  and the scalar product in the final term include integrals over energy and three-momenta as well as sums over the different types of field. The final term in the equation appears when we include fields that depend explicitly on the scale  $k$ , as in Ref. [28, 32].

For our regulator, we use the form suggested by Litim [33], which is optimised for local interactions. This suppresses the contributions of modes with momenta  $q < k$  by replacing their kinetic energy with the constant  $k^2/2m$ . For the atom field it has the form

$$R_\psi(q, k) = \frac{k^2 - q^2}{2m} \theta(k - q). \quad (2)$$

The dimer and trimer regulators have similar forms but also contain the wave-function renormalisation factors defined below.

The key ingredient in any practical application of the FRG is the choice of truncation of the running effective action. Here we work with the local form

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$$\begin{aligned} \Gamma_k[\psi, \psi^*, \phi, \phi^*, \chi, \chi^*] = \int d^4x \left[ \psi^* \left( i \partial_0 + \frac{\nabla^2}{2m} \right) \psi + Z_d \phi^* \left( i \partial_0 + \frac{\nabla^2}{4m} \right) \phi + Z_t \chi^* \left( i \partial_0 + \frac{\nabla^2}{6m} \right) \chi \right. \\ - u_d \phi^* \phi - u_t \chi^* \chi - \frac{g}{2} (\phi^* \psi \psi + \psi^* \psi^* \phi) - h (\chi^* \phi \psi + \phi^* \psi^* \chi) - \lambda \phi^* \psi^* \phi \psi \\ - \frac{u_{dd}}{2} (\phi^* \phi)^2 - \frac{v_d}{4} (\phi^* \phi^* \phi \psi \psi + \phi^* \psi^* \psi^* \phi \phi) - \frac{w}{4} \phi^* \psi^* \psi^* \phi \psi \psi \\ \left. - u_{tt} \chi^* \psi^* \chi \psi - \frac{u_{dt}}{2} (\phi^* \phi^* \chi \psi + \chi^* \psi^* \phi \phi) - \frac{v_t}{2} (\phi^* \psi^* \psi^* \chi \psi + \chi^* \psi^* \phi \psi \psi) \right]. \quad (3) \end{aligned}$$


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The running effective action in Eq. (3) contains kinetic terms for atom, dimer and trimer fields with wave-function renormalisation factors and interaction terms with up to four underlying atoms. The four-atom couplings without trimer fields ( $u_{dd}$ ,  $v_d$  and  $w$ ) were used by Schmidt and Moroz in their treatment of bosonic systems [30] and the analogous fermionic ones were studied in Ref. [31].

The wave-function renormalisation factors  $Z_{d,t}$ , self-energies  $u_{d,t}$  and couplings  $h$ ,  $\lambda$  etc. all run with the regulator scale  $k$ . In vacuum, there is no renormalisation factor for the atom field  $\psi$  and the coupling  $g$  remains constant during the evolution. We keep quadratic terms up to first order in time-derivatives, corresponding to linear dependences on energy. To second-order, spatial derivatives appear as required by Galilean invariance, as result of our choice of regulator [33].

The local form for the action reduces the functional differential equation (1) to a set of coupled ordinary differential equations for the coupling constants and renormalisation factors. Large numbers of diagrams contribute to

the driving terms, as in the versions without trimer fields studied in Refs. [30, 31]. Details of the equations will be presented elsewhere [34]; for now we concentrate on the salient features of their solutions.

We make a Hubbard-Stratonovich transformation to eliminate the atom-atom contact interaction at some large starting scale  $K$ . This term is not regenerated by the evolution and so atom-atom scattering is mediated only by the coupling  $g$  to dimers. At zero energy, the scattering is given by  $g^2/u_d(k)$  where  $u_d(k)$  evolves linearly with  $k$ . We choose its initial value such that, in the physical limit,  $u_d(0)$  gives the desired scattering length  $a$  [25, 26].

In contrast, the atom-dimer interaction,  $\lambda$ , is regenerated, even if we set it to zero initially. By introducing fields that depend explicitly on the scale  $k$ , as in Ref. [28, 32], we can cancel the evolution of some couplings. If we set their initial values to zero at the starting scale  $K$ , these couplings are effectively eliminated from the problem. Nonetheless their effects are implicitly present through additional terms that appear in the

flow equations for the remaining couplings. Here we take the trimer to run as

$$\partial_k \chi = \zeta_a \phi \psi + \zeta_b \psi^\dagger \chi \psi + \zeta_c \psi^\dagger \phi \phi + \zeta_d \psi^\dagger \phi \psi \psi, \quad (4)$$

where the quantity  $\zeta_a(k)$  cancels the running of  $\lambda$ . The other terms cancel the running of the four-atom couplings  $v_d$ ,  $w$  and  $v_t$ , as we discuss below.

For regulator scales  $k \gg 1/a$ , the quantities  $h^2(k)$ ,  $u_t(k)$  and  $Z_t(k)$  all have the form of powers of  $k$  times functions that oscillate periodically in  $t = \ln(k/K)$ . Zero-energy atom-dimer scattering is given by the combination  $h(k)^2/u_t(k)$ , which evolves in exactly the same way as  $\lambda(k)$  in the theory without the trimer [28, 30]. It displays poles that are equally spaced in  $t$  and reflect the passage of three-body bound states through zero energy as  $k$  is lowered. In the physical limit these build up the infinite tower of Efimov states. One three-body parameter is needed to fix the position of one of these poles, all others are then related to it by the Efimov scaling factor. With the truncated action used here the scaling factor in momentum is 29.8 [28, 30], which yields longer cycles than the exact value of 22.7 [5, 6].

In the four-atom sector, we eliminate the couplings  $v_d$ ,  $w$  and  $v_t$  that include the dimer-atom-atom channel. This leaves only the couplings involving the dimer-dimer and atom-trimer channels,  $u_{dd}$ ,  $u_{dt}$  and  $u_{tt}$ . This choice reflects the structure of the Faddeev-Yakubovsky equations used in most direct calculations of four-body systems [35]. In contrast, Schmidt and Moroz [30] also introduced a trimer field to treat energy dependence but used it to leave only the couplings  $u_{dd}$ ,  $v_d$  and  $w$ .

The evolution of the four-atom couplings is governed by a system of three coupled nonlinear differential equations. The appearance of  $h^2(k)$ ,  $u_t(k)$  and  $Z_t(k)$  in them means that they inherit the Efimov periodicity of the three-body sector. It also leads to two types of singularity in the coefficients in these equations. The most important have denominators with either one or two powers of

$$E_a^R(k) + E_t^R(k) = \frac{k^2}{2m} + \frac{k^2}{6m} + \frac{u_t(k)}{Z_t(k)}, \quad (5)$$

where  $E_{a,t}^R$  are the regulated atom and trimer energies. These are points at which an atom-trimer threshold passes through zero energy as we lower  $k$ . At each of them we expect additional contributions to the imaginary parts of the four-body couplings, as a channel with a new Efimov state becomes open.

The other type of divergent term has a factor of  $1/(h(k))^2$ . These lead to unphysical singularities in the four-body couplings, which mark the start of a short region within each Efimov cycle where  $h^2(k)$  and  $Z_t(k)$  have opposite signs. In these regions, the trimer field has a ghost-like character, with a propagator  $h^2(k)/(Z_t p_0 - u_t(k))$  that has a negative residue at its pole. This is a

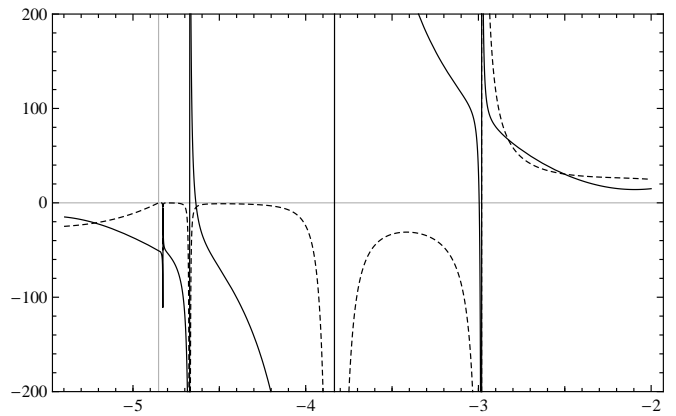


FIG. 1. One Efimov cycle of the flow of the rescaled coupling  $k^{406/125} u_{tt}(k)$  plotted against  $t = \ln(k/K)$ . The real part is shown by the dashed curve and the imaginary part by the solid one. The atom-trimer threshold corresponding to the vanishing of Eq. (5) is marked by the grey vertical line at  $t = t_3 \simeq -4.85$ .

warning that not all features of the effective action are physical for non-zero values of  $k$ . Fortunately these regions are well separated from the threshold regions where the phenomena of interest occur.

We have numerically integrated the coupled equations for  $u_{dd}(k)$ ,  $u_{dt}(k)$  and  $u_{tt}(k)$  through several Efimov cycles, and we have checked that any transients caused by our choice of initial conditions die out within the first cycle. The results for one cycle of the coupling  $u_{tt}(k)$  in the unitary limit are shown in Fig. 1. At the value of  $t = \ln(k/K)$  where the atom-trimer threshold passes through zero energy,  $t = t_3 \simeq -4.85$ , we see the expected discontinuity in the slope of the imaginary part signalling the opening of a new channel. The unphysical singularity arising from the zero of  $h^2(k)$  is the structure that can be seen at  $t \simeq -3.0$ .

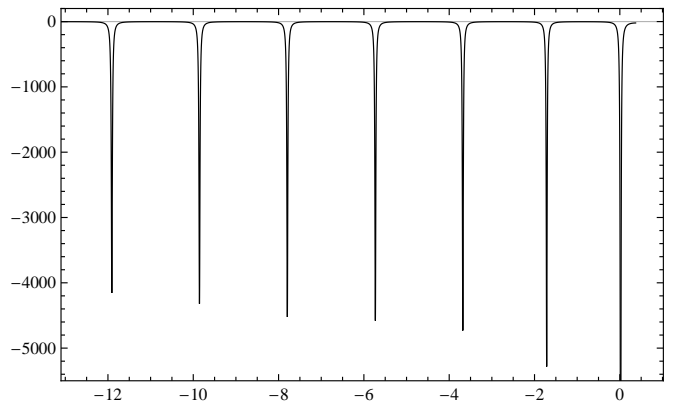


FIG. 2. The imaginary part of  $(t - t_3)^{-1} k^{406/125} u_{tt}(k)$  just before a threshold, plotted against  $x = \ln(t - t_3)$ . Apart from the rightmost one, corresponding to the deepest four-body state, the poles are approximately equally spaced.

Several simple poles can also be seen in Fig. 1, at  $t \simeq -3.83$ ,  $-4.67$ , and just below the threshold. When we look more closely at the region close to an atom-trimer threshold, as in Fig. 2, we find an infinite sequence of these poles. These become equally spaced in the variable  $x = \ln(t - t_3)$ . These poles do not correspond to singularities in the equations but are generated by the evolution of the couplings. Like the singularities that appear in the three-body sector, we interpret them as bound states or, rather, narrow resonances since they have finite imaginary parts as a result of coupling to open channels with more deeply bound trimers.

The introduction of the trimer field to describe energy dependence in the three-body sector is essential for generating these states as they do not appear in the FRG equations for the couplings without trimer fields [36].<sup>1</sup> The scales at which these states appear follow a double-exponential, “super-Efimov” pattern, similar to that observed in the two-dimensional three-body system studied by Moroz *et al.* [22].

Mathematically this structure arises from the forms of our differential equations which are analogous to that of the RG equation of Ref. [22]. The key terms that lead to the “super-Efimov” behaviour are the ones that are singular at the atom-trimer threshold. These arise from diagrams that are similar to those in Fig. 2 of that paper. However we should stress these states appear for non-zero values of  $k$ , where the action is not physical. Moreover the four-body flow equations depend on a scale as a result of the breaking of scale invariance by the Efimov effect. These states may therefore move relative to the atom-trimer threshold during the evolution to the physical limit. If so, only a finite number of bound states may persist in that limit.

The local form of the action, Eq. (3), does not allow us to study the full energy dependence in the four-body channels and so we cannot directly determine the spectrum in the physical limit. Instead, we can examine where these states cross zero energy as we move away from the unitary limit by taking a non-zero atom-atom scattering length,  $a < 0$ . Such zero-energy states are the ones observed in experiments on ultra-cold atoms in traps, as they lead to resonant enhancements of the loss of atoms at particular values of the scattering length [10, 18].

With a finite scattering length, the final Efimov cycle no longer has the same form as in the unitary limit. An example is shown in Fig. 3. For  $t \gtrsim -2.3$  the flow of the four-atom coupling matches Fig. 1, but beyond this point differences become increasingly visible. The example shown has the scattering length tuned so that the

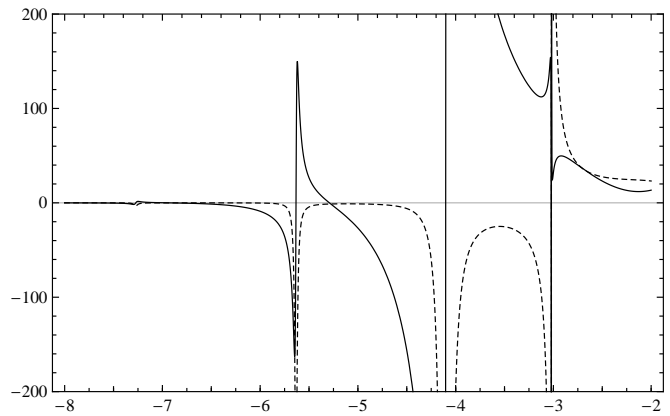


FIG. 3. The final cycle of the flow of the rescaled coupling  $k^{406/125} u_{tt}(k)$  plotted against  $t = \ln(k/K)$ . The atom-atom scattering length has been tuned so that the last three-body state appears at  $k = 0$  ( $t = -\infty$ ).

shallowest trimer state has exactly zero energy at  $k = 0$ . In this case, we find three four-body states appearing in the final Efimov cycle (the poles close to  $t = -4.1$ ,  $-5.6$  and  $-7.1$ ). There is thus no conflict with the theorem of Amado and Greenwood [37] that there are only a finite number of these four-body states. We denote the corresponding scattering length by  $a_3$ . When we further decrease  $a$ , we find that the values  $a_4^{(n)}$  at which the four-body states cross zero energy are related to  $a_3$  by

$$a_4^{(0)}/a_3 \simeq 0.438, \quad a_4^{(1)}/a_3 \simeq 0.877, \quad a_4^{(2)}/a_3 \simeq 0.9967. \quad (6)$$

For the two lowest states, these ratios are within 5% of the results of exact solutions to the four-body equations [13, 38], and hence they are also in reasonable agreement with the experimental numbers [18]. The third state lies extremely close to the atom-trimer threshold. If it is real, then it will be a challenge to observe both numerically and experimentally. However this state may just be an artefact of our truncation since improvements to the action which shorten the Efimov cycle might make it unbound.

Returning to the double-exponential behaviour observed during the evolution, the scale  $k_4^{(n)}$  at which the  $n$ -th excited four-body state appears can be written in the form

$$k_4^{(n)} = k_3 \exp[\alpha e^{-\beta n}], \quad (7)$$

where  $\alpha \simeq 1.53$ ,  $\beta \simeq 2.06$ , and  $k_3$  denotes the scale corresponding to the atom-trimer threshold for the next three-body Efimov state. This describes the energies of all states except the lowest ( $n = 0$ ) to a very good approximation. The ratios between scales for subsequent states can be expressed in the form of a universal scaling function,

$$k_4^{(n+1)}/k_4^{(n)} = \left(k_3/k_4^{(n)}\right)^{1-e^{-\beta}}. \quad (8)$$

<sup>1</sup> The four-body states seen in Ref. [30] have been found to be numerical artefacts [36].

This is similar to the scaling relation between the binding energies found by Hadizadeh *et al.* [15, 16], although its detailed functional form is different. More importantly, and in contrast to the results of those authors, we do not find an independent four-body scale: our parameter  $\alpha$  has a fixed value which is independent of the initial conditions we impose on the four-body couplings. This independence of any four-body parameter also applies to the physical states discussed above.

In summary: we have used the FRG to study systems of four bosons close to the unitary limit. In contrast to previous approaches, we introduce a trimer field to treat energy dependence of three-body subsystems and to reflect the channel structure of the Faddeev-Yakubovsky equations. In the physical limit, where the cut-off scale tends to zero, we examine the points at which three- and four-body states pass through zero energy as we vary the atom-atom scattering length. We find three four-body states in the last Efimov cycle of the three-body physics. The lowest two of these pass through zero for scattering lengths that are in good agreement with the results of exact solutions of the Faddeev-Yakubovsky equations [13, 38] and with experimental observations [18]. The third state is extremely weakly bound and may be an artefact of our truncated action.

In the unitary limit, the evolution generates an infinite number of four-body resonant states during each Efimov cycle, although it seems unlikely that all of these persist to the physical limit. These states lie just below each atom-trimer threshold and follow a double-logarithmic, or “super-Efimov” pattern [22]. They obey a universal scaling relation analogous to that of Ref. [15]. However the scales at which they appear are independent of the initial conditions on the four-body couplings. This supports the conclusion of Refs. [12, 39] that there is no additional relevant parameter in four-boson systems with contact interactions.

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